Note 174: Abstract Implication Semantics

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Hi all,
I think this is something for next week or later, but I wanted to share it now, in case anyone is thinking about it (as I suggested to Dan to think about it). So please ignore this note for November 28, 2022, unless you are independently interested.
Best,
Ulf

I continue the line of thought from two weeks ago. I would like to formulate our implication-space semantics in terms of inferential roles, rather than in terms of sentences. My hope is to achieve (a) a cleaner separation of contents and their bearers, (b) a formulation that is as much as possible neutral with respect to and independent of any particular content bearers, (c) an account on which we can say that reason-relations (including logical relations) hold among contents and not just among sentences, and (d) a formulation of the semantic clauses that looks clean and slick.

1 Abstract Implication Semantics

Now, I don't (yet) see how to do without any content bearers at all. But I hope that we can use them to climb up to the level of abstract contents. Ultimately, I hope that we will be able to be "functionalists" regarding content bearers and say: whatever play such-and-such a role is a content bearer with this-and-that content. This would be a top-down story. But, for now, the story is still bottom-up, from content-bearers to contents.

Definition 1 (Simply content bearers and their implication space)**.** The members of the set $\texttt{SCB}$ are the simple content bearers. And all the pairs of subsets of $\texttt{SCB}$, i.e. $\mathcal{P}(\texttt{SCB})^2$, is the bearer-implication-space, which contains all the candidate implications among content bearers.

Definition 2 (Content bearer implication)**.** Content bearer implication is a relation between sets of simple content bearers, $\mathbb{I} \subseteq \mathcal{P}(\texttt{SCB})^2$.

Definition 3 (Content-bearer-implications-equivalence, $\approx$)**.** Content-bearer-implications-equivalence, $\approx$, is a relation between sets of content bearer candidate implications, namely: let G and F be sets of content bearer candidate implications, then $G \approx F$ iff $\forall \langle X, Y \rangle$ $(\forall \langle \Pi, \Sigma \rangle \in F \ \langle X \cup \Pi, Y \cup \Sigma \rangle \in \mathbb{I}$ iff $\forall \langle \Gamma, \Psi \rangle \in G \ (\langle X \cup \Gamma, Y \cup \Psi \rangle \in \mathbb{I}))$.

Proposition 4. *Content-bearer-implications-equivalence is an equivalence relation.*

Proof. Obvious, but let's be explicit. The relation, $\approx$, is reflexive because $\forall \langle X, Y \rangle$ $(\forall \langle \Pi, \Sigma \rangle \in F \ \langle X \cup \Pi, Y \cup \Sigma \rangle \in \mathbb{I}$ iff $\forall \langle \Gamma, \Psi \rangle \in F \ (\langle X \cup \Gamma, Y \cup \Psi \rangle \in \mathbb{I}))$. The relation is symmetric because "iff" is commutative. To see that $\approx$ is transitive suppose that $G \approx F$ and $H \approx F$. So for any $\langle X, Y \rangle$ we have $\forall \langle \Pi, \Sigma \rangle \in G \ \langle X \cup \Pi, Y \cup \Sigma \rangle \in \mathbb{I}$ iff $\forall \langle \Gamma, \Psi \rangle \in F \ \langle X \cup \Gamma, Y \cup \Psi \rangle \in \mathbb{I})$ iff $\forall \langle \Pi, \Sigma \rangle \in H \ \langle X \cup \Pi, Y \cup \Sigma \rangle \in \mathbb{I}$. Therefore, any $\langle X, Y \rangle$ we have $\forall \langle \Pi, \Sigma \rangle \in G \ \langle X \cup \Pi, Y \cup \Sigma \rangle \in \mathbb{I}$ iff $\forall \langle \Pi, \Sigma \rangle \in H \ \langle X \cup \Pi, Y \cup \Sigma \rangle \in \mathbb{I}$. So $G \approx H$. ■

Definition 5 (Implication Roles)**.** The implication roles, $\mathcal{R}$, are the equivalence classes of sets of content bearer implications under the equivalence $\approx$. That is, $R \in \mathcal{R}$ iff $\exists G \subseteq \mathcal{P}(\texttt{SCB})^2$ $(R = \{x \mid G \approx x\})$.

We have now reached the level of pure contents, relative to a space of content bearers.

Definition 6 (Content)**.** A content, $c \in \mathcal{C}$, is a pair of implication roles, i.e., $\mathcal{C} = \mathcal{R} \times \mathcal{R}$, and if $c = \langle R_1, R_2 \rangle$, we say that R_1 is the premisory role of c and R_2 is its conclusory role.

The following definition will not be used below and is hence superfluous. But to see what is going on, it may be helpful to realize that the just given definition of contents is a generalization of our familiar definition, which we may call "simple contents."

Definition 7 (Simple Content)**.** A simple content, $c \in \mathcal{SC}$, is a pair of implication roles $\langle r^+, r^- \rangle$ such that $\exists p \in \texttt{SCB}$ such that $r^+ = \{x \mid \langle p, \emptyset \rangle \approx x\}$ and $r^- = \{x \mid \langle \emptyset, p \rangle \approx x\}$.

Note that if there is a countable infinity of simply contents (and hence elements of SCB), then there is an uncountable infinity of contents. So any language induces a space of contents that is larger than its set of distinct sentences. In other words, for simple Cantorian (pigeonhole) reasons, we can never make all contents of our language explicit in the form of claimables.

We can now formulate a notion of reason-relation, which I call "entailment," that holds between inferential roles, i.e. abstract contents, and not among sentences.

Definition 8 (Content Entailment, $\Vdash$)**.** Let $\Gamma, \Delta \subseteq \mathcal{C}$, then $\Gamma \Vdash \Delta$ iff for all $\langle \gamma_i^+, \gamma_i^- \rangle \in \Gamma$ and $\left\langle \delta_j^+, \delta_j^- \right\rangle \in \Delta\ \exists \langle g_i^+, g_i^- \rangle \subseteq \mathcal{P}(\texttt{SCB})^2$ ($\gamma_i^+ = \{x \mid \langle g_i^+, g_i^- \rangle \approx x\}$) and[1] $\exists \left\langle d_j^+, d_j^- \right\rangle \subseteq \mathcal{P}(\texttt{SCB})^2$ ($\delta_j^- = \{x \mid \left\langle d_j^+, d_j^- \right\rangle \approx x\}$) such that $\left\langle \bigcup g_i^+ \cup \bigcup d_j^+, \bigcup g_i^- \cup \bigcup d_j^- \right\rangle \in \mathbb{I}$.

To see what's going on, note that we adjoin some representative of all the premisory roles for each premise and the conclusory roles for each conclusion and check whether this adjunction of representatives of the roles is a good content bearer implication.

[1]Notice that the pairs of sets of content bearers are each only one representative from, respectively, the positive and negative roles of the contents in Γ and Δ respectively.

At this point, you may worry that different representatives of the roles will yield difference results when we check for entailment between their roles. Fortunately, that isn't the case.

Proposition 9. *It doesn't matter which element of the equivalence classes one picks, i.e., if one of their combinations is in $\mathbb{I}$, then so are all of them.*

Proof. Suppose that for all $\langle \gamma_i^+, \gamma_i^- \rangle \in \Gamma$ and$\left\langle \delta_j^+, \delta_j^- \right\rangle \in \Delta$ $\exists \langle g_i^+, g_i^- \rangle \subseteq \mathcal{P}(\texttt{SCB})^2$ $(\gamma_i^+ = \{x \mid \langle g_i^+, g_i^- \rangle \approx x\})$ and $\exists \left\langle d_j^+, d_j^- \right\rangle \subseteq \mathcal{P}(\texttt{SCB})^2$ $(\delta_i^- = \{x \mid \left\langle d_j^+, d_j^- \right\rangle \approx x\})$ such that $\left\langle \bigcup g_i^+ \cup \bigcup d_j^+, \bigcup g_i^- \cup \bigcup d_j^- \right\rangle \in \mathbb{I}$. And let $\langle \bar{g}_k^+, \bar{g}_k^- \rangle \in \gamma_k \in \Gamma$. And consider let α be $\left\langle \bigcup g_i^+ \cup \bigcup d_j^+, \bigcup g_i^- \cup \bigcup d_j^- \right\rangle$ with g_k^+ replaced by $\bar{g}_k^+$ and g_k^- replaces by $\bar{g}_k^-$ is not in $\mathbb{I}$. Since $\langle \bar{g}_k^+, \bar{g}_k^- \rangle \in \gamma_k$, we know that $\langle X \cup \{\bar{g}_k^+\}, Y \cup \{\bar{g}_k^-\} \rangle \in \mathbb{I}$ iff $\forall \langle \Gamma, \Psi \rangle \in \gamma_k (\langle X \cup \Gamma, Y \cup \Psi \rangle \in \mathbb{I})$. But we also know that $\langle g_k^+, g_k^- \rangle \in \gamma_k$. So $\left\langle \bigcup g_i^+ \cup \bigcup d_j^+, \bigcup g_i^- \cup \bigcup d_j^- \right\rangle$ is such an instance of this for $\Gamma = g_k^+$ and $\Psi = g_k^-$ and $X = \bigcup g_i^+ \cup \bigcup d_j^+ \setminus g_k^+ \cup g_k^-$. So that, in our case, $\langle X \cup \{\bar{g}_k^+\}, Y \cup \{\bar{g}_k^-\} \rangle$ is just α. Therefore $\alpha \in \mathbb{I}$ iff $\left\langle \bigcup g_i^+ \cup \bigcup d_j^+, \bigcup g_i^- \cup \bigcup d_j^- \right\rangle \in \mathbb{I}$. Hence, α must also be in $\mathbb{I}$. This generalizes to simultaneous replacements of several representatives of equivalence classes. ■

Definition 10 (Adjunction)**.** There is an associative and commutative operation on $\mathcal{R}$ known as adjunction, "$\sqcup$". For singleton sets of implications, $R_1 = \{x \mid \{\langle \Gamma, \Theta \rangle\} \approx x\}$ and $R_2 = \{x \mid \{\langle \Delta, \Lambda \rangle\} \approx x\}$, $R_1 \sqcup R_2 =_{df.} \{x \mid \{\langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle\} \approx x\}$. We also generalize $\sqcup$ as an operation over sets of implications as follows: let $F, G \subseteq \mathcal{P}(\texttt{SCB})^2$ and let $R_1 = \{x \mid F \approx x\}$ and $R_2 = \{x \mid G \approx x\}$, then: $R_1 \sqcup R_2 = \{x \mid x \approx \{\{f\} \sqcup \{g\} | f \in F, g \in G\}\}$.

This is basically our usual definition of adjunction, but everything is lifted to the level of roles, i.e. equivalence classes of implications. As for entailment, we an show that it doesn't matter which representative of an equivalence class you use to determine an adjunction: they all yield the same result.

Proposition 11. *Adjunction has unique results, i.e., no matter which element of an equivalence class one picks to calculate the result of adjoining the equivalence classes, the result is the same equivalence class of sets of implications.*

Proof. We take the singleton case first. Suppose that $R_1 = \{x \mid \{\langle \Gamma, \Theta \rangle\} \approx x\}$ and $R_2 = \{x \mid \{\langle \Delta, \Lambda \rangle\} \approx x\}$, and let $\langle A, B \rangle \in R_1$ and $\langle C, D \rangle \in R_2$. It suffices to show that $\{x \mid \{\langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle\} \approx x\} = \{x \mid \{\langle A \cup C, B \cup D \rangle\} \approx x\}$, for which it in turn suffices to show that $\{\langle A \cup C, B \cup D \rangle\} \approx \{\langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle\}$. We know that $\{\langle A, B \rangle\} \approx \{\langle \Gamma, \Theta \rangle\}$ and $\{\langle C, D \rangle\} \approx \{\langle \Delta, \Lambda \rangle\}$. Hence, for any pair $\langle X, Y \rangle$ we have $\langle X \cup A, Y \cup B \rangle \in \mathbb{I}$ iff $\langle X \cup \Gamma, Y \cup \Theta \rangle \in \mathbb{I}$; and $\langle X \cup C, Y \cup D \rangle \in \mathbb{I}$ iff $\langle X \cup \Delta, Y \cup \Lambda \rangle \in \mathbb{I}$. So, $\langle X \cup \Gamma \cup \Delta, Y \cup \Theta \cup \Lambda \rangle \in \mathbb{I}$ iff $\langle X \cup A \cup \Delta, Y \cup B \cup \Lambda \rangle \in \mathbb{I}$ iff $\langle X \cup A \cup C, Y \cup B \cup D \rangle \in \mathbb{I}$. Therefore, $\{\langle A \cup C, B \cup D \rangle\} \approx \{\langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle\}$. This shows that Adjunction is unique for the singelton case.

For the case of sets of implications, we can be quicker and less explicit. Let $F' \approx F$. It suffices to show that $\{x \mid x \approx \{\{f\} \sqcup \{g\} | f \in F, g \in G\}\} = \{x \mid x \approx \{\{f'\} \sqcup \{g\} | f' \in F', g \in G\}\}$. For this it suffices to show that $\forall \langle X, Y \rangle$ we have $\forall \langle f^+, f^- \rangle \in F(\langle X \cup f^+, Y \cup f^- \rangle \in \mathbb{I}$ iff $\forall \langle f'^+, f'^- \rangle \in F'(\langle X \cup f'^+, Y \cup f'^- \rangle \in \mathbb{I}$. But that is precisely what is ensured by $F' \approx F$. ∎

In our usual setting, we use intersections among $\vee$-sets. Since we are now working at the level of roles, we must lift this operation on implications to an operation on roles, which I will call "symjunction" (though I don't like that name, I just needed something that isn't already taken).

Definition 12 (Symjunction)**.** There is an associative and commutative operation on R known as symjunction, "$\sqcap$". Let $F, G \subseteq \mathcal{P}(\texttt{SCB})^2$ and let $R_1 = \{x \mid F \approx x\}$ and $R_2 = \{x \mid G \approx x\}$, then: $R_1 \sqcap R_2 = \{y \mid \forall z \in \{x \mid \forall f \in F(f \sqcup x \in \mathbb{I})\} \cap \{x \mid \forall g \in G(g \sqcup x \in \mathbb{I})\}(z \sqcup y \in \mathbb{I})\}$.

Note: The definition of symjunction may seem complicated, but all it does is to isolate the part of two inferential roles that they share. All of the things that yield a good implication when adjoint with any element of one of the two roles yields a good implication when adjoint to their symjunction.

Proposition 13. *Symjunction has unique results, i.e., no matter which element of an equivalence class one picks to calculate the result of symjoining the equivalence classes, the result is the same equivalence class of sets of implications.*

Proof. Let $F' \approx F$. Then $\forall f \in F(f \sqcup x \in \mathbb{I})$ iff $\forall f' \in F'(f' \sqcup x \in \mathbb{I})$. Similarly for another representative of the inferential role $R_2 = \{x \mid G \approx x\}$. ■

Once we have abstract contents as well as the adjunctions and symjunctions of their premisory and conclusory roles, we can define interpretations of a language as functions that assign sentences contents. And we can do this in a clean and slick way for logically complex languages.

Definition 14 (Interpretation Function)**.** An interpretation function $[\![\cdot]\!]$ is defined inductively, and maps sentences of a language $\mathfrak{L}$ to contents in models. If $A \in \mathfrak{L}$ is atomic, then $[\![A]\!] =_{df.} \langle a^+, a^- \rangle \in \mathcal{C}$. The connective clauses are as follows:

$[\![A \& B]\!] =_{df.} \langle a^+ \sqcup b^+, a^- \sqcap b^- \rangle$,
$[\![A \vee B]\!] =_{df.} \langle a^+ \sqcap b^+, a^- \sqcup b^- \rangle$,
$[\![A \to B]\!] =_{df.} \langle a^- \sqcap b^+, a^+ \sqcup b^- \rangle$,
$[\![\neg A]\!] =_{df.} \langle a^-, a^+ \rangle$.

Interpretations of sets of sentences are the set of the interpretants of the sentences, i.e., $\Gamma = [\![G]\!] = \{[\![g_i]\!] \mid g_i \in G\}$.

Definition 15 (Models)**.** A model, $\mathcal{M}$, is a pair $\left\langle \mathcal{C}, [\![\cdot]\!]^{\mathcal{M}} \right\rangle$ consisting of a set of contents and an interpretation function $[\![\cdot]\!]^{\mathcal{M}}$ that maps sentences to these contents.

Definition 16 (Linguistic Entailment)**.** We say that sentences G linguistically entail sentences D, on model $\mathcal{M}$, iff the corresponding entailment holds among their interpretants, $\Gamma = [\![G]\!]^{\mathcal{M}}$ and $\Delta = [\![D]\!]^{\mathcal{M}}$, on that model, i.e., iff $\Gamma \Vdash \Delta$.

2 Why Should We Do This?

Moving to the level of abstract inferential roles may seem like a needless complication that doesn't yield any benefits. But I think this isn't the case. Here are some reasons why:

1. We now have objects that are contents, pairs of premisory and conclusory roles. Different sentences (or even collections of sentences, or implications) can have the same content.

2. We can define reason relations among contents, rather than just among the sentences that express contents.

3. We can make sense of the case in which we don't have a content bearer that plays a certain role but would like to have one. There will always be contents that are not expressed by any single content bearer.

4. We have a notion of intra-language and inter-language synonymy: Two sentences are synonymous iff they express the same content.

5. We need some space of content bearers to climb to the space of abstract contents. But once we are in the space of abstract contents, we can use them to interpret other content bearers.

3 Dot-Quotes

I think that the content of a content bearer is very similar to what Sellars expressed by dot-quotes. Sellars deals with subsentential expressions, but nothing prevents us from dot-quoting sentences. So, •*Es regnet*• is a distributive singular term that allows us to talk collectively about the expressions, in any language, that play the same role (in their language) that "*Es regnet*" plays in German. For example, the following is true: "It is raining" is a •*Es regnet*•. And: The •*Es regnet*• is an assertoric sentence. And:

•*Es regnet*•s are present tense sentences. And: The •*Es regnet*• is the •*It is raining*•.

Now, our interpretation function works a bit differently. Like dot-quotes, $[\![\cdot]\!]$ creates a singular term. We can, e.g., say $[\![\text{Es regnet}]\!] = [\![\text{It is raining}]\!]$. But our interpretation function is not nominalistic, and Sellars invented dot-quotes chiefly for the purpose of defending his nominalism. For, the interpretants of sentences are abstracta. And interpretants are not distributive singular terms. So we cannot say: "It is raining" is a $[\![\text{Es regnet}]\!]$. We must say: The content of "It is raining" is $[\![\text{Es regnet}]\!]$. And we cannot say: The $[\![\text{Es regnet}]\!]$ is an assertoric sentence. We must say: $[\![\text{Es regnet}]\!]$ is expressed by assertoric sentences.

I think, however, that these differences are the result of Sellars's nominalism. He didn't want to be committed to universals or abstract individuals. And dot-quotes were supposed to help him in this endeavor. But if we divide through by this metaphysical goal of Sellars's, then dot-quotes do two things: (a) they group together expressions of the same language that are equivalent in their meaning-relevant behavior, and (b) they do the same for expressions of different languages.

Now, we think that the meaning-relevant behavior of a sentence consists in the role it can play as a premise and the role it can play as a conclusion. So to sentences are equivalent in their meaning-relevant behavior if they have the same premisory and conclusory roles. If $\exists p \in \texttt{SCB}$, then we can (ad a) group together expressions from the same space of content bearers by saying that they are all the sentences whose content is $\langle r^+, r^- \rangle$, where $r^+ = \{x \mid \langle p, \emptyset \rangle \approx x\}$ and $r^- = \{x \mid \langle \emptyset, p \rangle \approx x\}$. These are exactly the expressions with equivalent behavior as premises and conclusions. And (ad b) we can do exactly the same for contents from different spaces of content bearers, i.e., from different languages. We must have to say that their interpretations are the same content.